

Connectivity and W_v -Paths in Polyhedral Maps on Surfaces

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This paper is dedicated to the memory of Victor Klee.

Abstract

The W_v -Path Conjecture due to Klee and Wolfe states that any two vertices of a simple polytope can be joined by a path that does not revisit any facet. This is equivalent to the well-known Hirsch Conjecture. Klee proved that the W_v -Path Conjecture is true for all 3-polytopes (3-connected plane graphs), and conjectured even more, namely that the W_v -Path Conjecture is true for all general cell complexes. This general W_v -Path Conjecture was verified for polyhedral maps on the projective plane and the torus by Barnette, and on the Klein bottle by Pulapaka and Vince. Let G be a graph polyhedrally embedded in a surface Σ , and x, y be two vertices of G . In this paper, we show that if there are three internally disjoint (x, y) -paths which are homotopic to each other, then there exists a W_v -path joining x and y . For every surface Σ , define a function $f(\Sigma)$ such that if for every graph polyhedrally embedded in Σ and for a pair of vertices x and y in $V(G)$, the local connectivity $\kappa_G(x, y) \geq f(\Sigma)$, then there exists a W_v -path joining x and y . We show that $f(\Sigma) = 3$ if Σ is the sphere, and for all other surfaces $3 - \tau(\Sigma) \leq f(\Sigma) \leq 9 - 4\chi(\Sigma)$, where $\chi(\Sigma)$ is the Euler characteristic of Σ , and $\tau(\Sigma) = \chi(\Sigma)$ if $\chi(\Sigma) < -1$ and 0 otherwise. Further, if x and y are not cofacial, we prove that G has at least $\kappa_G(x, y) + 4\chi(\Sigma) - 8$ internally disjoint W_v -paths joining x and y . This bound is sharp for the sphere. Our results indicate that the W_v -path problem is related to both the local connectivity $\kappa_G(x, y)$, and the number of different homotopy classes of internally disjoint (x, y) -paths as well as the number of internally disjoint (x, y) -paths in each homotopy class.

Keywords: W_v -path Conjecture, polyhedral embedding, homotopy class, local connectivity

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1 Introduction

The W_v -Path Conjecture (or *Non-revisiting Path Conjecture*), originally due to Klee and Wolfe (cf. [8]), states that any two vertices of a simple polytope P can be joined by a path that does not revisit

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any facet of P . (Such a non-revisiting path is also called a W_v -path.) Klee further conjectured that the W_v -Path Conjecture is true for general cell complexes [10]. Larman [13] showed that this general W_v -Path Conjecture is false for a very general type of 2-dimensional complex and later Mani and Walkup [14] found a 3-sphere counterexample. The original W_v -Path Conjecture for boundary complexes of polytopes is known to be equivalent to two other well-known conjectures, the *Hirsch Conjecture* and the *Dantzig d -step Conjecture*, involving higher dimensional polytopes which in turn are important in the continuing search for a practical polynomial algorithm for the simplex method of linear programming. For proofs of these equivalences we direct the reader to [8, 9] and [11]. The Dantzig d -step Conjecture was verified by Klee and Walkup for all bounded polyhedra for $d \leq 5$ [12]. In 2012, the Hirsch Conjecture was shown to be false by Santos (cf. [21].)

The first positive result related to the general W_v -Path Conjecture was obtained by Klee [8] who showed that every pair of vertices of a 3-connected plane graph G (or “3-polytope”) are joined by a W_v -path. (See also [6] and Grünbaum [7].) One of the nice properties of 3-connected plane graphs is that their faces meet “properly”. (Here and throughout the rest of the paper we consider a face to include its boundary.) That is, they meet at a single vertex, a single edge or not at all. This idea has been generalized to surfaces other than the plane by the notion of a *polyhedral embedding*. An embedding of a graph G in a surface Σ is *polyhedral* if every face is a closed disk and any two faces of the embedding meet properly, which is equivalently to saying the representativity (face-width) of the embedding is at least 3 (cf. [15]). It follows that a graph admitting a polyhedral embedding must be 3-connected (cf. [15]).

The general W_v -Path Conjecture has also been studied for polyhedral embeddings of graphs in general 2-dimensional surfaces as well. (Here by “2-dimensional surface” we mean a connected compact 2-manifold without boundary.) The W_v -Path Conjecture in this context states that for every surface (orientable or non-orientable) and every graph polyhedrally embedded therein, there is a W_v -path joining every pair of distinct vertices. Klee’s result on 3-connected plane graphs was later extended to graphs polyhedrally embedded in the projective plane [2] and torus [3] by Barnette, and in the Klein bottle by Pulapaka and Vince [18]. It is now known, however, that the W_v -Path Conjecture is false for every orientable surface of genus $g \geq 2$ and for every non-orientable surface of genus $\bar{g} \geq 4$ (cf. [17]). Hence the sole unsettled case is the non-orientable surface with $\bar{g} = 3$. For a summary of these results, see [4, 16, 17, 18]. As positive results for the W_v -Path Conjecture are rare, the departure point in the present paper is an attempt to ascertain what conditions suffice to make the W_v -Path Conjecture hold.

Let G be a polyhedrally embedded graph in a surface Σ . Given two distinct vertices x and y in a graph G , they are *cofacial* if they belong to the boundary of a common face. If the cofacial vertices x and y are adjacent, then there is exactly one W_v -path joining them (the single edge xy). If they are not adjacent, there are exactly two W_v -paths joining them, namely the two paths forming the boundary of the face. In this paper, we will focus on the case in which x and y are *non-cofacial*. The *local connectivity* $\kappa_G(x, y)$ of two vertices x and y is defined to be the maximum number of internally disjoint paths joining x and y , where two paths joining x and y are *internally disjoint* if they have only x and y in common. A graph G is *k -connected* if $\kappa_G(x, y) \geq k$ for any two vertices x and y . We observe that the W_v -path problem is closely related to both the local connectivity

$\kappa_G(x, y)$, and the number of homotopy classes of (x, y) -paths as well as the number of (x, y) -paths in each homotopy class. In order to describe our results, define $f(\Sigma)$ to be the smallest value such that for every graph G polyhedrally embedded in the surface Σ and for a pair of vertices x and y of G , if $\kappa_G(x, y) \geq f(\Sigma)$, then there exists a non-revisiting (x, y) -path. The following is one of our main results, in which $\chi(\Sigma)$ denotes the Euler characteristic of the surface Σ .

Theorem 1.1. *Let Σ be a closed surface. Then $f(\Sigma) = 3$ if Σ is the sphere. For all other surfaces $3 - \tau(\Sigma) \leq f(\Sigma) \leq 9 - 4\chi(\Sigma)$, where $\tau(\Sigma) = \chi(\Sigma)$ if $\chi(\Sigma) < -1$ and 0 otherwise.*

The lower bound is obtained by construction. In order to verify the upper bound, we introduce the concept of dual curve for revisits, which turns out to be very useful in bounding the number of revisits and the number of different homotopy classes of (x, y) -paths. In particular, we prove the following result.

Theorem 1.2. *Let G be a graph polyhedrally embedded in a surface Σ , and x and y be two non-cofacial vertices. If there exist three internally disjoint (x, y) -paths which are homotopic to each other, then there exists a non-revisiting (x, y) -path.*

The above result says three internally disjoint homotopic (x, y) -paths implies the existence of one non-revisiting path. However, Theorem 1.1 indicates that, for each surface Σ with Euler characteristic $\chi(\Sigma) < -1$, a graph G polyhedrally embedded in Σ may not have W_v -path between two vertices x and y if there are less than $3 - \chi(\Sigma)$ paths joining them. This shows that the non-revisiting path problem is related to the homotopy classes of (x, y) -paths.

Another application of our method provides a very short proof for the upper bound for the face touching number of a 3-connected graph embedded in a surface, which was originally proved by Sanders [20] using a discharging argument.

Besides the existence of W_v -paths, Barnette [1] also generalized the W_v -path result for the plane in a different direction. He proved that if two vertices of a graph polyhedrally embedded in the plane are non-cofacial, then they are joined by at least three internally disjoint W_v -paths. Richter and Vitray [19] proved that, in fact, if a graph is embedded in any surface with representativity at least 4, there are at least two internally disjoint homotopic W_v -paths joining any two non-cofacial vertices. In this paper, we also derive the following new relationship between the number of internally disjoint W_v -paths joining two vertices x and y and the local connectivity $\kappa_G(x, y)$.

Theorem 1.3. *Let G be a graph polyhedrally embedded in a surface Σ , and x and y be two non-cofacial vertices. Then G has at least $\kappa_G(x, y) + 4\chi(\Sigma) - 8$ internally disjoint non-revisiting (x, y) -paths.*

The bound in Theorem 1.3 is sharp for the sphere. This will be proved in Section 3. An even better bound for the projective plane will be given in Section 4.

2 Dual curves and contractible revisits

We begin with some definitions and notation. Let G be a graph polyhedrally embedded in a surface Σ and let x and y be two vertices of G . Let P be a path joining x and y . A face F is *revisited* by

the path P if $F \cap P$ has at least two components. Let $c(F \cap P)$ to be the number of components of $F \cap P$. The *total revisit number* of P is $r_P = \sum_F (c(F \cap P) - 1)$. Let S_1, S_2, \dots, S_k be the connected components of $F \cap P$. Throughout the paper, S_i for some integer i always stands for a connected component of the intersection of an (x, y) -path and some face. A pair $\{S_i, S_j\}$ is called a *revisit* to F by P . (See Figure 1 (left) where P is represented by the thick edges joining x and y and F is the face exterior to the outside octagon.)

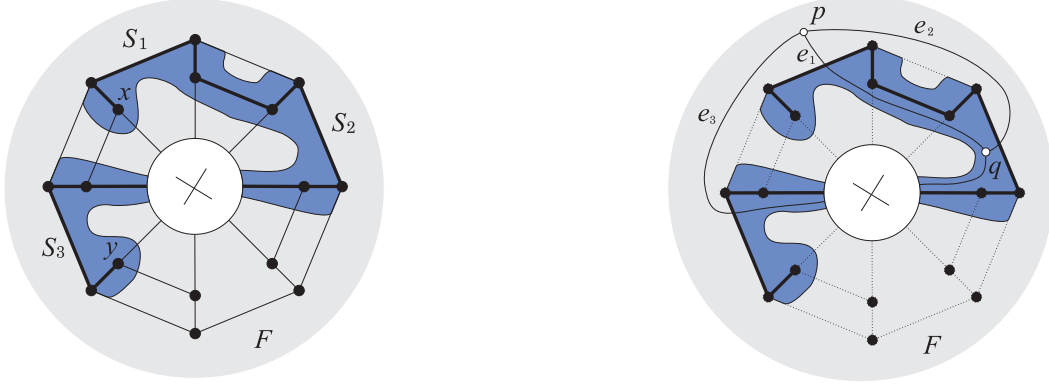


Figure 1: Dual curves, contractible and non-contractible revisits.

For an (x, y) -path P joining x and y , let $N_\epsilon(P)$ be a closed ϵ -neighborhood of P such that $N_\epsilon(P)$ is homotopic to a disk. Denote the interior of F by $\text{Int}(F)$. Let $N_{\text{top}}(P) = N_\epsilon(P) \setminus \text{Int}(F)$. Then $N_{\text{top}}(P)$ is homeomorphic to a closed disk. (See the dark grey area in Figure 1.)

Let p and q be two points lying in the interiors of F and $N_{\text{top}}(P)$ respectively. We construct an auxiliary graph $H(p, q)$ also embedded in Σ such that $H(p, q)$ has two vertices p and q and k edges $e_i, 1 \leq i \leq k$, where e_i joins p and q through S_i . Let $\ell_{ij} = e_i \cup e_j$. Then ℓ_{ij} is a simple closed curve which we will call the *dual closed curve* of the revisit $\{S_i, S_j\}$. In general, of course, a dual closed curve can be contractible or non-contractible. A revisit $\{S_i, S_j\}$ is *non-contractible* if ℓ_{ij} is non-contractible, and *contractible*, otherwise. For example, see Figure 1 (right) in which ℓ_{13} and ℓ_{23} are non-contractible and ℓ_{12} is contractible. Therefore $\{S_1, S_2\}$ is a contractible revisit, but $\{S_1, S_3\}$ is a non-contractible revisit in Figure 1 (left). Additional definitions and notation will be introduced below as needed.

Lemma 2.1. *Let G be a 3-connected graph embedded in a surface Σ and let x and y be two vertices of G . Let P be an (x, y) -path revisiting a face F such that every revisit is non-contractible. Then the number of components of $P \cap F$ is at most $4 - 2\chi(\Sigma)$.*

Proof. Assume that $P \cap F = \{S_1, \dots, S_k\}$ where $k \geq 2$. Let $H = H(p, q)$ be the auxiliary graph defined as above, namely let p and q be two vertices belonging to the interiors of F and $N_{\text{top}}(P)$ respectively, and suppose, for $i = 1 \dots k$, e_i is an edge joining p and q through S_i . Then the graph H is embedded in Σ . Since every revisit $\{S_i, S_j\}$ is non-contractible, every dual closed curve $\ell_{ij} = e_i \cup e_j$ is non-contractible. If $e_i \cup e_j$ bounds a face of H , then the interior of the face is not homomorphic to an open disk. In this case, cut the face along $e_i \cup e_j$ and cap off its boundary curve $e_i \cup e_j$. For

every face of size 2, apply this operation so that eventually we generate a new surface Σ' such that every face of size 2 of H embedded in Σ' has its interior homomorphic to an open disk. Let f_2 be the number of faces of H with size 2. Then $\chi(\Sigma') \geq \chi(\Sigma) + f_2$. Consider the graph H embedded in the surface Σ' . Then H has f_2 faces of size 2 which are closed disks. Let $F(H)$ be the set of all faces of H . Then by Euler's formula,

$$2 - |E(H)| + |F(H)| \geq \chi(\Sigma')$$

where equality holds if the interior of every face is homomorphic to an open disk. Let f_{4+} be the number of faces of H with size at least 4, then $|F(H)| = f_2 + f_{4+}$. Hence

$$2 - k + (f_2 + f_{4+}) \geq \chi(\Sigma) + f_2.$$

It follows that $\chi(\Sigma) \leq 2 - k + f_{4+}$. Note that $2k = 2|E(H)| \geq 2f_2 + 4f_{4+} \geq 4f_{4+}$, and further, $f_{4+} \leq k/2$. Combining this inequality with $\chi(\Sigma) \leq 2 - k + f_{4+}$, it then follows that $k \leq 4 - 2\chi(\Sigma)$. \square

The *face touching number* of two faces F_1 and F_2 is the number of components of $F_1 \cap F_2$. The face touching number of a graph G is the maximum face touching number over all pairs of faces of G . Assume that $F_1 \cap F_2 = \{S_1, \dots, S_k\}$ where S_i is a connected component of $F_1 \cap F_2$. If G is 3-connected, every component S_i is a single edge or vertex and hence the boundary of F_1 contains at least one edge xy which is not on the boundary of F_2 . Then deleting xy from the boundary of F_1 results in a path, which we will denote by P . Note that $P \cap F_2 = F_1 \cap F_2$ as $xy \notin F_1 \cap F_2$. By the 3-connectivity of G , we can conclude that every revisit $\{S_i, S_j\}$ to F_2 by P is non-contractible. Otherwise, $\{S_i, S_j\}$ contains a 2-vertex-cut of G as the dual closed curve ℓ_{ij} of $\{S_i, S_j\}$ is contractible and hence separating, a contradiction to the 3-connectivity of G . By Lemma 2.1, we have the following result on face touching numbers of 3-connected graphs, which was originally proved by Sanders [20] using a discharging argument.

Corollary 2.2 ([20]). *Let G be a 3-connected graph embedded in a surface Σ . Then the face touching number of G is at most $4 - 2\chi(\Sigma)$.*

Remark. The bound of Lemma 2.1 is tight in that equality may hold. Sanders constructed examples to illustrate that the face touching number of a 3-connected graph can reach $4 - 2\chi(\Sigma)$. Again, if one traverses a path P from the boundary of one of the two faces of the examples of Sanders, then P revisits the other face $4 - 2\chi(\Sigma)$ times.

The following lemma gives a condition under which the number of contractible revisits can be reduced.

Lemma 2.3. *Let G be a graph polyhedrally embedded in a surface Σ , and x and y be two non-cofacial vertices. Suppose $\mathcal{P} = \{P_1, \dots, P_k\}$ ($k \geq 3$) is a set of k internally disjoint (x, y) -paths. If a face F has a contractible revisit by path P_i , there exists a path P'_i such that $\mathcal{P}' = (\mathcal{P} \setminus \{P_i\}) \cup \{P'_i\}$ is a set of k internally disjoint (x, y) -paths with $r_{\mathcal{P}'} < r_{\mathcal{P}}$.*

Proof. Assume that $F \cap P_i = \{S_1, S_2, \dots, S_t\}$. Without loss of generality, we may assume that $\{S_1, S_2\}$ is a contractible revisit. Then the dual curve ℓ_{12} bounds a disk D . If the disk D contains any other component of $F \cap P_i$, say S_j , then the dual curve ℓ_{1j} bounds another disk D' which is contained inside D . Since the number of revisits is finite, there exists a revisit such that its dual curve bounds a disk which does not contain any other revisits. Therefore, without loss of generality, assume that D does not contain any other component of $F \cap P_i$.

Let v_1, v_2 be two endvertices of S_1 and u_1, u_2 be two endvertices of S_2 such that v_1, v_2, u_1 and u_2 appear on the boundary of F in clockwise order. Note that, it is possible that $v_1 = v_2$ and/or $u_1 = u_2$. Assume that the segment of the boundary of F inside the disk D from v_2 to u_1 is denoted by v_2Fu_1 .

If one of x and y is outside the disk D and the other is inside the disk D , then an (x, y) -path $P_j \in \mathcal{P}$ with $j \neq i$ will intersect the boundary of disk D by the Jordan Curve Theorem; in other words, P_j intersects P_i , contradicting the fact that P_i and P_j are internally disjoint. Hence x and y are either both outside the disk D or both inside the disk D .

First, assume that both x and y are outside the disk D . Then every $P_j \in \mathcal{P}$ with $j \neq i$ is disjoint from v_2Fu_1 . Further, assume that P_i is traversed from x to v_1 first and then v_2 . By the definition of dual curve, the segment of path P_i from S_1 to S_2 together with v_2Fu_1 forms a curve homotopic to ℓ_{12} . Since y is outside D , it follows that P_i passes through u_1 first and then u_2 . Let $v_2P_iu_1$ stand for the subpath of P_i joining v_2 and u_1 , and let $P'_i = (P_i \setminus v_2P_iu_1) \cup v_2Fu_1$. Then P'_i is internally disjoint from $P_j \in \mathcal{P}$ with $j \neq i$. Since y is outside of the disk D , the segment of $P_i \setminus v_2P_iu_1$ from u_2 to y does not intersect the cycle $v_2Fu_1 \cup v_2P_iu_1$. Note that every face visited by v_1Fu_1 , except F , lies inside the disk bounded by $v_2Fu_1 \cup v_2P_iu_1$. It then follows that v_1Fu_1 does not revisit any other face F' , for if there were such a revisit, the two faces F and F' would touch twice, contradicting the fact that G is polyhedrally embedded in Σ . So $r_{\mathcal{P}'_i} < r_{\mathcal{P}}$.

So in the following, assume that both x and y are inside the disk D . Then all other (x, y) -paths $P_j \in \mathcal{P}$ with $j \neq i$ are inside D , for otherwise, P_j intersects P_i , a contradiction of the fact that P_i and P_j are internally disjoint. Now let $P'_i = (P_i \setminus v_1P_iu_2) \cup u_2Fv_1$. Then P'_i is disjoint from P_j since u_2Fv_1 is outside D . Any face F' visited by u_2Fv_1 is outside D and is not visited by the segments $P_i \setminus v_1P_iu_2$ which are inside D . So u_1Fv_1 does not revisit any face of G since G is polyhedrally embedded in Σ . Therefore $r_{\mathcal{P}'_i} < r_{\mathcal{P}}$. This completes the proof. \square

If G is polyhedrally embedded in the plane, then every revisit to a face by a path is contractible. So the following result, which strengthens a classical result of Barnette on W_v -paths ([1]), is an immediate corollary of Lemma 2.3.

Theorem 2.4. *Let G be a graph polyhedrally embedded in the sphere and x, y two non-cofacial vertices of G . Then there are at least $\kappa_G(x, y)$ internally disjoint W_v -paths joining x and y .*

3 Polyhedral maps on general surfaces

In this section, we will prove our main results, namely Theorems 1.1, 1.2 and 1.3.

Let x and y be two vertices of a graph G polyhedrally embedded in a surface Σ . Two internally disjoint (x, y) -paths P and P' are *homotopic* to each other if $P \cup P'$ bounds an open disk of Σ . Given a family \mathcal{P} of internally disjoint (x, y) -paths, a *homotopy class* \mathcal{P}' of \mathcal{P} is a subfamily of \mathcal{P} such that any two paths of \mathcal{P}' are homotopic to each other and any path $P \in \mathcal{P} \setminus \mathcal{P}'$ is not homotopic to any path in \mathcal{P}' . Note that, if Σ is the sphere, then all internally disjoint (x, y) -paths are homotopic to each other and hence there is exactly one homotopy class of any given family of internally disjoint (x, y) -paths in this case.

Lemma 3.1. *Let G be a connected graph embedded in a surface Σ different from the sphere, and suppose $x, y \in V(G)$. Then the number of homotopy classes of a family of internally disjoint (x, y) -paths is no more than $4 - 2\chi(\Sigma)$.*

Proof. Let \mathcal{P} be a family of internally disjoint (x, y) -paths and let k be the total number of homotopy classes of \mathcal{P} . Since Σ is not the sphere, $\chi(\Sigma) < 2$. If $k = 1$, then the lemma holds trivially. So in the following, suppose that $k \geq 2$. Choose one (x, y) -path P_i ($i = 1, 2, \dots, k$) from each homotopy class. Then no two of P_1, \dots, P_k are homotopic to each other.

We construct an auxiliary graph H embedded in Σ as follows: let $V(H) = \{x, y\}$ and $E(H) = \{e_1, \dots, e_k\}$ where e_i is a single edge joining x and y and is homotopic to path P_i . Then H is a bipartite multigraph with two vertices and k edges. Since P_i is not homotopic to P_j for $j \neq i$, the same is true for e_i and e_j . Therefore, $e_i \cup e_j$ is a non-contractible cycle of H . If $e_i \cup e_j$ bounds a face, then the interior of the face is not homomorphic to an open disk. An argument similar to that used in the proof of Lemma 2.1 shows that $k = |E(H)| \leq 4 - 2\chi(\Sigma)$. \square

The following result illustrates an important connection between homotopy classes and W_v -paths.

Lemma 3.2. *Let G be a graph polyhedrally embedded in a surface Σ , and let x and y be two non-cofacial vertices of G . Let \mathcal{P} be a homotopy class of a family of internally disjoint (x, y) -paths of G , and assume that D is the minimal disk containing all paths \mathcal{P} . Then D contains at least $|\mathcal{P}| - 2$ internally disjoint non-revisiting (x, y) -paths.*

Proof. Assume that $\mathcal{P} = \{P_1, \dots, P_k\}$ is a homotopy class of a family of internally disjoint (x, y) -paths. If $k \leq 2$, the lemma holds trivially. So suppose that $k \geq 3$. As D is the minimal disk containing all paths in \mathcal{P} , we can conclude that D is bounded by two paths in \mathcal{P} , say P_1 and P_k .

Note that all (x, y) -paths contained in D are homotopic. We choose a set of k internally disjoint (x, y) -paths in D , denoted by $\mathcal{P}' = \{P'_1, \dots, P'_k\}$, such that the total revisit number of \mathcal{P}' is minimal. Relabeling if necessary, we may assume that all paths in \mathcal{P}' are contained in a disk bounded by P'_1 and P'_k . Every revisit to a face F in D by an (x, y) -path in \mathcal{P}' is contractible. By Lemma 2.3 and the choice of \mathcal{P}' , all paths in \mathcal{P}' except P'_1 and P'_k , are W_v -paths joining x and y . It then follows immediately that D contains at least $|\mathcal{P}| - 2$ non-revisiting (x, y) -paths. \square

Theorem 1.2 follows immediately from Lemma 3.2. Now, we are going to prove Theorem 1.3.

Proof of Theorem 1.3. Assume that $\kappa_G(x, y) = k$. Then G has k internally disjoint (x, y) -paths. Assume these k internally disjoint (x, y) -paths can be partitioned into t homotopy classes $\mathcal{P}_1, \dots, \mathcal{P}_t$.

It follows from Lemma 3.1, that $t \leq 4 - 2\chi(\Sigma)$. Let D_i be the minimal disk containing all paths in \mathcal{P}_i . By Lemma 3.2, each disk D_i contains at least $|\mathcal{P}_i| - 2$ internally disjoint non-revisiting (x, y) -paths. Therefore, the total number of internally disjoint non-revisiting (x, y) -paths is at least

$$\sum_{i=1}^t (|\mathcal{P}_i| - 2) = \sum_{i=1}^t |\mathcal{P}_i| - 2t \geq k + 4\chi(\Sigma) - 8.$$

□

Theorem 1.3 guarantees that if the local connectivity $\kappa_G(x, y)$ is large enough, then G has a non-revisiting (x, y) -path. It would be interesting to find the *minimum* local connectivity for graphs polyhedrally embedded in a surface Σ which guarantees the existence of a non-revisiting (x, y) -path. Define $f(\Sigma)$ to be the smallest number k such that for any graph G polyhedrally embedded in Σ and any two vertices x and y in G , if $\kappa_G(x, y) \geq k$, then G has at least one non-revisiting (x, y) -path. By Theorem 2.4, $f(\Sigma) = 3$ if Σ is the sphere. The results obtained in [8, 1, 2, 3, 18] show that $f(\Sigma) = 3$ for surfaces with $\chi(\Sigma) \geq 0$. For all other surfaces Σ , the exact value of $f(\Sigma)$ remains unknown. By Theorem 1.3, we have the following result which provides an upper bound of $f(\Sigma)$ for surfaces Σ with $\chi(\Sigma) < 0$.

Corollary 3.3. *Let Σ be a surface. Then $f(\Sigma) = 3$ if $\chi(\Sigma) \geq 0$ and $f(\Sigma) \leq 9 - 4\chi(\Sigma)$ otherwise.*

In the following, we are going to construct examples in which the lower bound given in Theorem 1.1 holds. Since G is polyhedrally embedded in a surface Σ , it follows that G is 3-connected and therefore $f(\Sigma) \geq 3$. The lower bound in Theorem 1.1 holds trivially for surfaces Σ with $\chi(\Sigma) \geq -1$. Now, we construct examples to illustrate the even better lower bound $f(\Sigma) \geq 3 - \chi(\Sigma)$ for surfaces with $\chi(\Sigma) < -1$, i.e., all orientable surfaces with genus $g \geq 2$ and non-orientable surfaces with genus $\bar{g} \geq 4$.

For each orientable genus $g \geq 2$ and each non-orientable genus $\bar{g} \geq 4$, we now exhibit examples of graphs with these genera having the property that they contain vertices x and y such that $\kappa_G(x, y) = 2g = 2 - \chi(\Sigma)$ in the orientable case and such that $\kappa_G(x, y) = \bar{g} = 2 - \chi(\Sigma)$ in the non-orientable case, but there is no W_v -path joining x and y .

First we consider the orientable case for all $g \geq 2$. Let x and y be two distinct vertices. Let $C_x = 11'22'33' \cdots nn'1$ be a cycle on $2n$ vertices such that $N(x) = \{1, 2, \dots, n\}$, where $n = 2g$. Similarly, let $C_y = 1(g+1)'2(g+2)' \cdots gn'(g+1)1' \cdots ng'1$ be a second $(2n)$ -cycle such that $N(y) = \{1', 2', \dots, n'\}$, again where $n = 2g$. For $i = 1, \dots, n$, join vertex i in C_x and $i \in C_y$ as well as i' in C_x and $i' \in C_y$. Call the resulting graph H_1 . (See Figure 2.) The cyclic orders of edges incident with vertices as shown in Figure 2 define a *rotation* scheme which represents an embedding of H_1 in an orientable surface Σ . By Euler's formula, the surface Σ has genus g . The faces of the embedding derived from the rotation system shown in Figure 2 are of the form $xi i'(i+1)x$, $yi i'(g+i+1)(i+1)'y$, and $ii' i'(g+i)(g+i)(g+i)'(g+i)'i$ where all integers are taken modulo n .

Now envision the graph H_1 embedded in this surface Σ . Next contract all edges of the form ii and $i'i'$. Call the resulting graph H_2 . Then H_2 inherits the embedding of H_1 in the surface Σ such that each facial 8-cycle in H_1 of the form $ii' i'(g+i)(g+i)(g+i)'(g+i)'i$ in H_1 corresponds to a facial

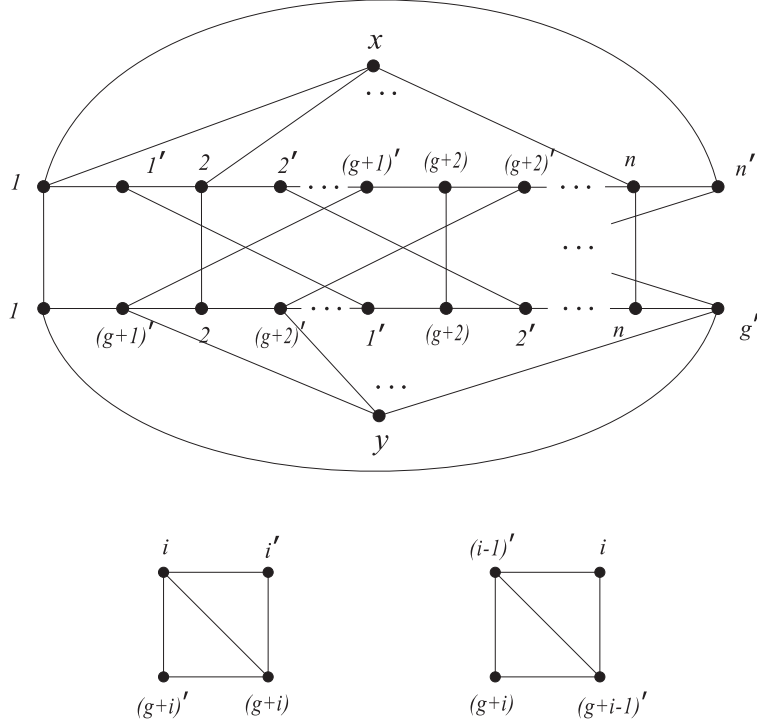


Figure 2: Orientable surfaces.

4-cycle $ii'(g+i)(g+i)'i$ in H_2 , and other facial 4-cycles of H_1 are still facial 4-cycles of H_2 . Now the graph H_2 is embedded in the surface Σ where every face is bounded by a 4-cycle. This embedding is not polyhedral because, for example, the 4-faces $ii'(g+i)(g+i)'i$ and $(i-1)'i(g+i-1)'(g+i)(i-1)'$ share vertices i and $(g+i)$ which are two components of the intersection of the face boundaries. So we add some additional diagonal edges to some of these paired 4-cycles as follows: for each $i = 1, \dots, n$, to the cycle $ii'(g+i)(g+i)'i$ we add the diagonal edge $i(g+i)$ and to the cycle $(i-1)'i(g+i-1)'(g+i)(i-1)'$ we add the diagonal edge $(i-1)'(g+i-1)'$. (See Figure 2.)

The resulting graph on $4g+2$ vertices, which we will call Γ_g , is then polyhedrally embedded in the orientable surface Σ of genus g and $\kappa_G(x, y) = 2g$. Note that an (x, y) -path starting with an edge xi revisits either a face incident with x (for example $x(g+i)(g+i)'(g+i+1)x$ or $x(g+i)(g+i-1)'(g+i-1)x$) or a face incident with y (for example $i(g+i)y(g+i-1)i$). So in Γ_g there are no W_v -paths joining x and y .

We now turn to the non-orientable case. For the non-orientable surface Σ where $\chi(\Sigma) = 2 - \bar{g}$ is even (i.e., $\bar{g} = 2k$), we proceed as follows.

Let x and y be distinct vertices, and $C_x = 11'22' \dots nn'1$ be a $2n$ -cycle with $N(x) = \{1, 2, \dots, n\}$ and let $C_y = 1(k+1)'2(k+2)' \dots kn'(k+1)1' \dots nk'1$ be a second $2n$ -cycle with $N(y) = \{1', 2', \dots, n'\}$ where $n = \bar{g}$. Join vertex i of C_x to i of C_y and vertex i' of C_x to i' of C_y . As in the orientable case, we also add all “vertical” edges of the form ii and $i'i'$ and call the resulting graph H_1 . This time, however, we position a separate crosscap on each the edges $1'1', 2'2', \dots, n'n'$ in H_1 to obtain

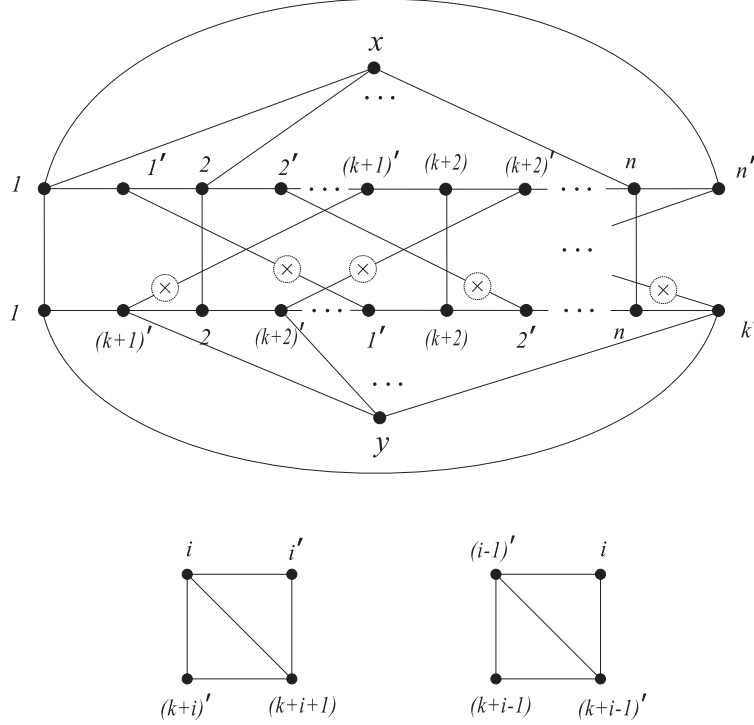


Figure 3: Non-orientable surfaces.

a non-orientable graph $\overline{H_1}$. The rotation scheme as shown in Figure 3 represents an embedding of H_1 in a non-orientable surface Σ . Again by Euler's formula, the surface Σ has non-orientable genus \bar{g} .

We contract all edges of the form ii and $i'i'$. We denote by $\overline{H_2}$ the resulting graph embedded in the surface Σ . In so doing, the 8-faces of the form $ii'i'(k+1+i)(k+1+i)(k+i)'(k+i)i$ and $i'(i+1)(i+1)(k+i)'(k+i)'(k+i)(k+i)i'$ contract to the 4-faces $ii'(k+1+i)(k+i)'i$ and $i'(i+1)(k+i)'(k+i)i'$ respectively. As before, we obtain pairs of quadrilaterals which share two vertices on their boundaries which are not consecutive on either boundary. So again we add the diagonal edges $i(k+i+1)$ and $(i-1)'(k+i-1)'$ to $\overline{H_2}$ to obtain a polyhedrally embedded graph which we shall call $\Gamma_{\bar{g}}$. In this embedded graph $\Gamma_{\bar{g}}$, $\kappa_{\overline{G}}(x, y) = \bar{g}$. Again, in $\Gamma_{\bar{g}}$, an (x, y) -path revisits either a face incident with x or a face incident with y . Therefore, there is no W_v -path joining x and y in $\Gamma_{\bar{g}}$.

We can modify the above construction for even non-orientable genera in order to treat the case when the non-orientable genus is *odd* as follows. Begin with the embedded graph $\overline{H_2}$ of even non-orientable genus \bar{g} and select any triangular face F . Denote it by $F = abca$. Now add two new adjacent vertices d and e and a new crosscap to the interior of F . Join a to d and e , b to e and c to d . Finally, join c to e and b to d through the crosscap. (See Figure 4.) The graph we seek is obtained from the original $\overline{H_2}$ by adding the new crosscap and the above seven new edges. This graph, then, has (odd) non-orientable genus $\bar{g} + 1$.

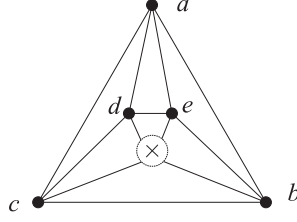


Figure 4: The added crosscap.

The above examples show that $k_G(x, y) = 2 - \chi(\Sigma)$ does not guarantee the existence of a W_v -path joining x and y . Therefore $f(\Sigma) \geq 3 - \chi(\Sigma)$ for surfaces Σ with $\chi(\Sigma) < -1$. By Corollary 3.3, Theorem 1.1 follows.

4 Polyhedral maps on the projective plane

In this section, we obtain a sharp lower bound for the number of internally disjoint non-revisiting (x, y) -paths for graphs polyhedrally embedded in the projective plane which improves the bound given in Theorem 1.3. Barnette's result [2] for the projective plane is a direct corollary of this result.

In the following, two closed curves α and β are *homotopically disjoint* if there exist two disjoint closed curves α' and β' such that α is homotopic to α' and β is homotopic to β' .

Theorem 4.1. *Let G be a graph polyhedrally embedded in the projective plane and suppose x and y are two non-cofacial vertices. Then there are at least $\kappa_G(x, y) - 2$ internally disjoint W_v -paths joining x and y .*

Proof. Let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a family of internally disjoint (x, y) -paths such that $r_{\mathcal{P}} = \sum_{i=1}^k r_{P_i}$ is minimum. If $r_{\mathcal{P}} = 0$, we are done, so in the following we will assume that $r_{\mathcal{P}} > 0$.

Suppose $P_1 \in \mathcal{P}$. Define \mathcal{P}_A by $\mathcal{P}_A = \{P_i \in \mathcal{P} | P_i \text{ is homotopic to } P_1\}$. Trivially, $P_1 \in \mathcal{P}_A$. Now define \mathcal{P}_B by $\mathcal{P}_B = \mathcal{P} - \mathcal{P}_A$. Then for any $P_i, P_j \in \mathcal{P}_A$, $P_i \cup P_j$ bounds a disk. Moreover, if $P_i \in \mathcal{P}_A$ and $P_\alpha \in \mathcal{P}_B$, $P_i \cup P_\alpha$ is a non-contractible cycle since P_i and P_α are not homotopic. Note that there is only one homotopy class of non-contractible simple closed curves on the projective plane since the fundamental group of this surface is \mathbb{Z}_2 . So all non-contractible cycles of G are homotopic. For $P_\alpha, P_\beta \in \mathcal{P}_B$, $P_1 \cup P_\alpha$ is homotopic to $P_1 \cup P_\beta$. It follows that P_α is homotopic to P_β . Hence \mathcal{P}_B is also a homotopy class of internally disjoint (x, y) -paths.

Without loss of generality, we may write $\mathcal{P}_A = \{P_1, \dots, P_t\}$ and $\mathcal{P}_B = \{P_{t+1}, \dots, P_k\}$, and also without loss of generality, we may assume that $|\mathcal{P}_A| \geq |\mathcal{P}_B|$. Note that $k \geq 3$ since G is 3-connected and \mathcal{P}_B may be empty. In any case $t \geq 2$.

Since \mathcal{P}_A is a homotopy class, $P_i \cup P_j$ bounds a disk, for any two distinct $P_i, P_j \in \mathcal{P}_A$. Therefore, all paths in \mathcal{P}_A are contained in a closed disk D bounded by the union of two paths in this set. Without loss of generality, let us renumber the paths if necessary, so that these two paths are denoted by P_1 and P_t . (See Figure 5 where the disk D is represented by the shaded region.) Similarly, we may suppose that paths P_{t+1} and P_k bound a closed disk D' containing all the paths in \mathcal{P}_B .

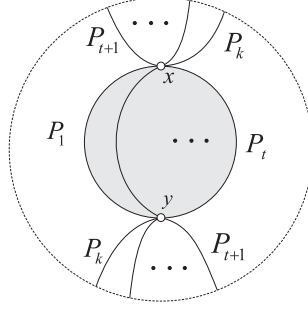


Figure 5: The two homotopy classes \mathcal{P}_A and \mathcal{P}_B of all (x, y) -paths.

By Lemma 2.3 and the minimality of $r_{\mathcal{P}}$, every revisit to any face F by an (x, y) -path in \mathcal{P} is non-contractible. So every face contained in the disk D bounded by $P_1 \cup P_t$ (respectively, in the disk D' bounded by $P_{t+1} \cup P_k$) is not revisited by any path in \mathcal{P} . Hence a face can only be revisited by P_1, P_t, P_{t+1} or P_k .

If $\mathcal{P}_B = \emptyset$, then a face can be revisited only by P_1 or $P_t (= P_k)$, so in this instance, there are at least $\kappa_G(x, y) - 2$ W_v -paths. So assume that $\mathcal{P}_B \neq \emptyset$. Let F be a face revisited by path P_1 . By Lemma 2.1, $F \cap P_1$ has exactly two components S_1 and S_2 .

Claim 1: *One of S_1 and S_2 is the single vertex x or y .*

Proof of Claim 1. Suppose to the contrary that $S_1 - \{x, y\}$ contains a vertex u and that $S_2 - \{x, y\}$ contains a vertex v . The dual closed curve ℓ_{12} of $\{S_1, S_2\}$ through u and v does not intersect $P_t \cup P_k$ which is a non-contractible cycle. Therefore, ℓ_{12} is contractible. Hence $\{S_1, S_2\}$ is a contractible revisit, a contradiction. This completes the proof of Claim 1.

If both homotopy classes \mathcal{P}_A and \mathcal{P}_B contain at most one path that is not a W_v -path, then trivially there are at least $\kappa_G(x, y) - 2$ W_v -paths. So in the following we will assume, without loss of generality, that class \mathcal{P}_A contains exactly two paths that are not W_v -paths, P_1 and P_t , since a face of G can only be revisited by P_1, P_t, P_{t+1} or P_k .

Claim 2: *The paths P_1 and P_t cannot revisit the same face.*

Proof of Claim 2: Suppose to the contrary that there exists a face F which is revisited by paths P_1 and P_t . By Lemma 2.1, $F \cap P_1$ has two components S_1 and S_2 . By Claim 1, one of S_1 and S_2 is the single vertex x or y . Suppose without loss of generality that $S_1 = \{x\}$. Similarly, $F \cap P_t$ has two components and one of them is the single vertex x or y . Since x and y are not cofacial, the vertex y cannot be a single vertex component of $F \cap P_t$. Therefore, $S_1 = \{x\}$ is also a component of $F \cap P_t$. Let S_3 be the other component of $F \cap P_t$.

Let ℓ_{12} and ℓ_{13} be the dual closed curves of $\{S_1, S_2\}$ and $\{S_1, S_3\}$ respectively. Note that both ℓ_{12} and ℓ_{13} are non-contractible. Therefore, ℓ_{12} and ℓ_{13} cross transversally at the vertex x . By the definition of dual closed curves, we assume that ℓ_{12} and ℓ_{13} intersect only at x (otherwise, other intersection components lie either in the face F or $N_{\text{top}}(P_1) \cap N_{\text{top}}(P_t)$, and hence can be contracted to x). Let $D'' = D \cup N_{\text{top}}(P_1) \cup N_{\text{top}}(P_t)$. Then the face F touches the disk D' four times along

ℓ_{12} and ℓ_{13} at $S_1 = \{x\}$, S_2 and S_3 . So the boundary of F self-intersects at x which contradicts the fact that G is polyhedrally embedded in the projective plane. This completes the proof of Claim 2.

By Claim 2, P_1 and P_t revisit two distinct faces F_1 and F_2 . By Lemma 2.1, $P_1 \cap F_1$ has exactly two components S_1^1 and S_2^1 and $P_t \cap F_2$ has exactly two components S_1^t and S_2^t . Next we show that both P_{t+1} and P_k are W_v -paths.

Assume there is a face F revisited by a path from \mathcal{P}_B , say P_k . Note that the boundary of F is homotopically disjoint from the boundary of $D \cup F_1 \cup F_2$, and therefore, the boundary of $D' \cup F$ is homotopically disjoint from the boundary of $D \cup F_1 \cup F_2$. Let ℓ_{12} be the dual closed curve of the revisits $\{S_1^1, S_2^1\}$ of F_1 by P_1 and ℓ' be the dual curve of the revisits of F by P_k . Therefore, ℓ_{12} and ℓ' are homotopically disjoint, a contradiction to the fact that both ℓ_{12} and ℓ' are non-contractible. This contradiction implies that P_k is a W_v -path. Similarly, so is P_{t+1} . It follows then that G contains at least $\kappa_G(x, y) - 2$ internally disjoint W_v -paths. \square

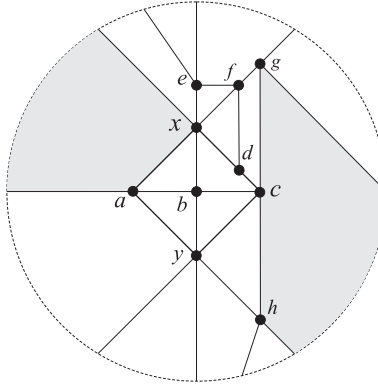


Figure 6: An example.

Remark: The example shown in Figure 6 shows that the bound of $\kappa_G(x, y) - 2$ in Theorem 4.1 for the projective plane is best possible. In the graph shown in this figure, there are six internally disjoint (x, y) -paths: xay , xby , $xdcy$, xey , $xfgy$ and xhy . Hence $\kappa_G(x, y) = 6$. But there are only four internally disjoint non-revisiting (x, y) -paths: xay , xby , xey and xhy as both $xdcy$ and $xfgy$ revisit the (shaded) face bounded by $axhcga$.

5 Concluding remarks

Let Σ be a closed surface and G be a graph polyhedrally embedded in Σ . A result of Cook [5] shows that the connectivity of G is at most $(5 + \sqrt{49 - 24\chi(\Sigma)})/2$ if $\chi(\Sigma) \leq 0$. It then follows that, if $\chi(\Sigma) < -7$, the connectivity of G is less than $3 - \chi(\Sigma)$. Then G may not have a W_v -path for some pair of vertices by Theorem 1.1. However, the locally connectivity of a pair of vertices x and y of G could be arbitrarily large. Hence, in the definition of $f(\Sigma)$, the local connectivity cannot be replaced by the connectivity of G .

Theorem 1.1 shows linear bounds for $f(\Sigma)$ for surfaces Σ . The previous results of Barnette [1, 2, 3] and Pulapaka and Vince [18] show that $f(\Sigma) = 3$ for surfaces with $\chi(\Sigma) \geq 0$. However, the exact values $f(\Sigma)$ for surfaces Σ with $\chi(\Sigma) < 0$ are unknown. It is interesting to ask the following question.

Problem 5.1. *Let Σ be a closed surface with $\chi(\Sigma) \leq -1$. Determine the exact value of $f(\Sigma)$.*

A solution to the above question would settle the existence problem of W_v -path in graphs polyhedrally embedded in the surface Σ with $\chi(\Sigma) = -1$, the only surface for which the existence of a W_v -path between a pair of vertices of G remains unknown.

Theorem 1.3 provides a lower bound for the number of internally disjoint W_v -paths between a pair of vertices of G . The bound is sharp for the sphere, but may not be sharp for other surfaces. Indeed, it is not tight for the projective plane. We propose the following.

Problem 5.2. *Let G be a graph polyhedrally embedded in the surface Σ and let x and y be two non-cofacial vertices. Find a sharp lower bound for the number of internally disjoint non-revisiting (x, y) -paths.*

Theorem 4.1 evidences that the number of internally disjoint non-revisiting (x, y) -paths is related to the Euler characteristic of the surface. But the connection is not clear. A solution to Problem 5.2 for the torus or the Klein bottle is interesting, which may lead to a complete solution to the problem.

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